E9 203 Compressed Sensing & Sparse SP

14 Apr 2021

Last time:

- Robust & stable recovery via NSP.

Today: Recovery of individual sparse vectors

Recall Thm. 2.16

For any $N \geq s+1$, given an s-sparse $x \in \mathbb{C}^N$, $\exists$ a meas. matrix $A \in \mathbb{C}^{m \times N}$, $m = s+1$, s.t. x can be reconstructed from $y = Ax$ as a soln. of (P_0): $\min_{z \in \mathbb{C}^N} \|z\|_0$ s.t. $Az = y$.

Defn. The sign of a (cplx.) number z:

$$\text{sgn}(z) = \begin{cases} \frac{z}{|z|}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Thm. 4.26: Given $A \in \mathbb{C}^{m \times N}$, the vec. $x \in \mathbb{C}^N$ with $\text{supp}(x) = S$ is the unique minimizer of $\|z\|_1$ s.t. $Az = Ax$ if one of the following equivalent conditions hold:

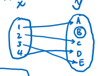
(a) $\left| \sum_{j \in S} \overline{\text{sgn}(x_j)} v_j \right| < \|v_{\bar{S}}\|_1 \quad \forall v \in \mathcal{N}(A) \setminus \{\underline{0}\}$.

(b) A_S is injective, & $\exists\, h \in \mathbb{C}^m$ s.t.

(b1): $(A^H h)_j = \text{sgn}(x_j)$, $j \in S$ ←

(b2): $|(A^H h)_\ell| < 1$, $\ell \in \bar{S}$. ←

(Injective: Distinct vecs. supported on S map to distinct obsrvn. vecs.



)

Proof: $\boxed{(a) \Rightarrow x \text{ is the unique} \ldots}$

Given $z \neq x$ s.t. $Az = Ax$, let $v \triangleq x - z$,

then $v \in \mathcal{N}(A) \setminus \{\underline{0}\}$

$z = x - v$
$z_{\bar{S}} = -v_{\bar{S}}$

$$\|z\|_1 = \|z_S\|_1 + \|z_{\bar{S}}\|_1$$

$$= \|(x-v)_S\|_1 + \|v_{\bar{S}}\|_1$$

$$> |\langle x - v, \text{sgn}(x)_S \rangle| + |\langle v, \text{sgn}(x)_S \rangle| \quad [\because (a)]$$

$$\geq |\langle x, \text{sgn}(x)_S \rangle| = \|x\|_1$$

$\|\tilde{x}\|_1 = \langle \tilde{x}, \text{sgn}(\tilde{x}) \rangle = (\text{sgn}(\tilde{x}))^H \tilde{x}$

Thus, (a) $\Rightarrow$ x unique min. of $\|z\|_1$ s.t. $Az = Ax$.

$\boxed{b \Rightarrow a}$: If $v \in \mathcal{N}(A) \setminus \{0\}$, $Av_S = -Av_{\bar{S}}$

$$\Rightarrow \left| \sum_{j \in S} \overline{\text{sgn}(x_j)} v_j \right| = |\langle v_S, A^H h \rangle| \quad [(b.1)]$$

$$= |\langle A^H v_S, h \rangle|$$

$$= |\langle A v_{\bar{S}}, h \rangle|$$

$$= |\langle v_{\bar{S}}, A^H h \rangle|$$

$$\leq \underbrace{\max_{\ell \in \bar{S}} |(A^H h)_\ell|}_{<1 \text{ by } [(b.2)]} \cdot \|v_{\bar{S}}\|_1 < \|v_{\bar{S}}\|_1$$

The last ineq. is strict $\because$

(i) $|(A^H h)_\ell| < 1$, $\ell \in \bar{S}$, and

(ii) $\|v_{\bar{S}}\|_1 > 0$. This is true $\because$ if $\|v_{\bar{S}}\|_1 = 0$ then $Av_S = 0$. $\Rightarrow$ $\exists$ vec. supported on S in $\mathcal{N}(A)$ which contradicts the assump. that A_S is injective.

$\boxed{(a) \Rightarrow (b)}$:

(a) $\Rightarrow$ $\|v_{\bar{S}}\|_1 > 0 \;\forall v \in \mathcal{N}(A) \setminus \{\underline{0}\}$

$\Rightarrow$ A_S is injective.

Consider $f(v) \triangleq \frac{|\langle v, \text{sgn}(x)_S \rangle|}{\|v_{\bar{S}}\|_1}$. }

When $v \in \mathcal{N}(A) \setminus \{\underline{0}\}$, $\|v\|_1 \leq 1$, $f(v) < 1$.

Further $f(v)$ is continuous, and $\|\cdot\|_1 < 1$ is a compact set. Hence

$$\max_{v \in \mathcal{B}} f(v) = \mu < 1.$$

$\mathcal{B}$: Unit ℓ_1-sphere $\cap\, \mathcal{N}(A)$

$$|\langle v, \text{sgn}(x)_S \rangle| \leq \mu \|v_{\bar{S}}\|_1 \quad \forall v \in \mathcal{N}(A)$$

Now define, for $\mu < \nu < 1$

$$\mathcal{C} \triangleq \{ z \in \mathbb{C}^N : \|z_S\|_1 + \nu \|z_{\bar{S}}\|_1 \leq \|x\|_1 \} \quad [\text{Convex}]$$

$$\mathcal{D} \triangleq \{ z \in \mathbb{C}^N : Az = Ax \} \quad [\text{Affine}]$$

Claim: $\mathcal{C} \cap \mathcal{D} = \{x\}$. Clearly, $x \in \mathcal{C} \cap \mathcal{D}$

If $z \neq x$, $z \in \mathcal{C} \cap \mathcal{D}$, $v = x - z \in \mathcal{N}(A) \setminus \{0\}$

$z = x - v$

$$\|x\|_1 \geq \|z_S\|_1 + \nu \|z_{\bar{S}}\|_1$$

$$= \|(x-v)_S\|_1 + \nu \|v_{\bar{S}}\|_1$$

$$> \|(x-v)_S\|_1 + \mu \|v_{\bar{S}}\|_1$$

$$\geq |\langle x - v, \text{sgn}(x)_S \rangle| + |\langle v, \text{sgn}(x)_S \rangle|$$

$$\geq |\langle x, \text{sgn}(x)_S \rangle| = \|x\|_1$$

i.e., a contradiction. Hence $\exists$ no other $z \in \mathcal{C} \cap \mathcal{D}$ besides x.

Use a result (Thm. B4): $\exists\, w \in \mathbb{C}^N$ s.t.

(*1) $\mathcal{C} \subset \{ z \in \mathbb{C}^N, \ \mathrm{Re}\langle z, w\rangle \le \|x\|_1 \}$.

(*2) $\mathcal{D} \subset \{ z \in \mathbb{C}^N, \ \mathrm{Re}\langle z, w\rangle = \|x\|_1 \}$.

Using (*1),

$$\|x\|_1 \ge \max_{\|z_S + \nu z_{\bar S}\|_1 \le \|x\|_1} \mathrm{Re}\langle z, w\rangle$$

$$= \max_{\|z_S + \nu z_{\bar S}\|_1 \le \|x\|_1} \mathrm{Re}\Big(\sum_{j \in S} z_j \bar w_j + \sum_{j \in \bar S} \frac{\nu z_j \bar w_j}{\nu} \Big)$$

$$= \max_{\dots} \mathrm{Re}\langle z_S + \nu z_{\bar S}, w_S + \tfrac{1}{\nu} w_{\bar S} \rangle$$

$$= \|x\|_1 \cdot \|w_S + \tfrac{1}{\nu} w_{\bar S}\|_\infty$$

$$= \|x\|_1 \max\{ \|w_S\|_\infty, \tfrac{1}{\nu}\|w_{\bar S}\|_\infty \}.$$

WLOG $x \ne 0$, so we get

$\|w_S\|_\infty \le 1$ and $\|w_{\bar S}\|_\infty \le \nu < 1$

From (*2), we get $\mathrm{Re}\langle x, w\rangle = \|x\|_1$

$\Rightarrow$ $w_j = \mathrm{sgn}(x_j) \ \forall j \in S$,

$\mathrm{Re}\langle v, w\rangle = 0 \ \forall v \in \mathcal{N}(A)$

i.e., $w \in \mathcal{N}(A)^\perp = \mathcal{R}(A^H)$

$\Rightarrow$ $w = (A^H h)$ for some $h \in \mathbb{C}^m$,

$(A^H h)_j = \mathrm{sgn}(x_j) \ \forall j \in S$

$|(A^H h)_\ell| < 1 \ \forall \ell \in \bar S$

That is, (a) $\Rightarrow$ (b), which completes the proof. $\square$